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The Problem of Identification of Parameters by the Distribution of the Maximum Random Variable

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Suppose that $X_1, X_2,..., X_n$ are independently distributed according to certain distributions. Does the distribution of the maximum of $\{X_1, X_2,..., X_n\}$ uniquely determine their distributions? In the univariate case, a general theorem covering the case of Cauchy random variables is given here. Also given is an affirmative answer to the above question for general bivariate normal random variables with non-zero correlations. Bivariate normal random variables with nonnegative correlations were considered earlier in this context by T. W. Anderson and S. G. Ghurye.

1. INTRODUCTION

If $F_1 F_2 \cdots F_n = G_1 G_2 \cdots G_m$, where the F_i's and G_i's are univariate (or all of them bivariate) distribution functions, then is $n = m$ and are $\{F_1, F_2,..., F_n\}$ a permutation of $\{G_1, G_2,..., G_m\}$? We consider this problem here. A general result covering the Cauchy distributions is given in Section 2. In Section 3, we solve this problem for bivariate normal distributions. For motivation and other preliminaries, we refer the reader to

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[1, 2]. In [1, 2], the bivariate normal case was only partly solved. Our proofs here are necessarily completely different from those in [1, 2].

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Consider the Cauchy distributions with parameters a_i's and b_j's given by

$$H_i(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(a_i x), \qquad L_j(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(b_j x),$$

where $-\infty < x < \infty$, $i = 1, 2,..., n$ and $j = 1, 2,..., m$. (Note that H_i can be expressed as $(1/\pi)(\pi/2 + \sum_{k=1}^{\infty}(-1)^k (a_i x)^k/k)$.) Suppose that $H_1 H_2 \cdots H_n = L_1 L_2 \cdots L_m$. Does it follow that $(H_i)_{i=1}^{n}$ is a rearrangement of $(L_j)_{j=1}^{m}$? We answer this below in a more general context.

First, we make the notations simpler and write

$$F_i(x) \equiv F(a_i x) \qquad \text{and} \qquad G_j(x) \equiv F(b_j x)$$

where

$$F(x) = x^t \cdot \sum_{p=0}^{\infty} k_p x^p, \qquad -a < x < a,\ a > 0,\ t \geqslant 0,\ k_0 \neq 0.$$

Suppose that

$$\prod_{i=1}^{n} F(a_i x) = \prod_{j=1}^{s} F(b_j x), \qquad -a < x < a. \tag{2.1}$$

We show below that $\{a_1, a_2,..., a_n\}$ is a permutation of $\{b_1, b_2,..., b_s\}$ under two different general conditions, each of which holds for the Cauchy distributions. The discussion covering the first condition is notationally tiresome; so the reader may prefer to go to the second condition (given in subsection B below) first.

A. Let us first present two simple lemmas. Besides being of indepedent interest, they will also be used in this part of the discussion. The proofs of these lemmas are not difficult and are omitted.

LEMMA 2.1. *Consider a function $f(x_1, x_2,..., x_n)$: $R^n \to R$ of the following form: $f = \sum x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_p}^{s_p}$, where p is a fixed positive integer ($\leqslant n$), $s_1, s_2,..., s_p$ are fixed positive integers such that $s_1 + s_2 + \cdots + s_p = d$ and the summation is over all possible p-tuples $(i_1, i_2,..., i_p)$ with the i_j's all different and $1 \leqslant i_j \leqslant n$. Then, f can be expressed as a polynomial with integer coefficients in the quantities $\sum_{i=1}^{n} x_i^j$, $j = 1, 2,..., d$. Moreover, in this polynomial, the term containing $\sum_{i=1}^{n} x_i^d$ is always $c \cdot \sum_{i=1}^{n} x_i^d$, where c is an integer.*

LEMMA 2.2. *Consider two finite sets of real numbers*

$$\{x_1, x_2,..., x_n\} \qquad \textit{and} \qquad \{y_1, y_2,..., y_n\}.$$

Suppose that

$$\sum_{i=1}^{n} x_i^k = \sum_{i=1}^{n} y_i^k, \qquad k = 1, 2,..., m \quad (m \leqslant n).$$

Then for $k = 1, 2,..., m$ the following equations also hold:

$$\sum x_{i_1} x_{i_2} \cdots x_{i_k} = \sum y_{i_1} y_{i_2} \cdots y_{i_k},$$

where the summations are taken as in Lemma 2.1 or over all possible k-tuples $(i_1, i_2,..., i_k)$ with $i_1 < i_2 < \cdots < i_k$ and $1 \leqslant i_j \leqslant n$ for each j.

Now we go back to (2.1). Expanding the products in (2.1), it is easily verified that the coefficient of x^{nt+r}, $1 \leqslant r \leqslant n$, on the left side of (2.1), is

$$\begin{aligned} &k_0^{n-r} k_1^r \cdot \sum_{i_1 < i_2 < \cdots < i_r} a_{i_1} a_{i_2} \cdots a_{i_r} \\ &+ \sum_{p=1}^{r-1} \sum_{s_1 + s_2 + \cdots s_p = r} \sum k_0^{n-p} k_{s_1} k_{s_2} \cdots k_{s_p} \cdot a_{i_1}^{s_1} \cdots a_{i_p}^{s_p}. \end{aligned} \tag{2.2}$$

(The third summation in the second term above is taken over all p-tuples $(i_1, i_2,..., i_p)$ such that the i_j's are all different, $1 \leqslant i_j \leqslant n$, and $i_{j_1} < i_{j_2}$ whenever $s_{j_1} = s_{j_2}$.)

Notice that by Lemma 2.1 (2.2) can be expressed as a polynomial in

$$\left\{ \sum_{i=1}^{n} a_i^j : j = 1, 2,..., r \right\},$$

where the coefficient of $\sum_{i=1}^{n} a_i^r$ is of the form

$$\begin{aligned} &C_{rr} = k_0^{n-r} [C_r + C_{r-1} k_0 + \cdots + C_1 k_0^{r-1}], \\ &C_r = k_1^r, \qquad C_j = \sum_{s_1 + s_2 + \cdots + s_j = r} c(s_1,..., s_j) k_{s_1} \cdots k_{s_j} \qquad (1 < j \leqslant r). \end{aligned}$$

(Here the c's are integers.)

Now we assume that $C_{rr} \neq 0$. (For Cauchy distributions, we can assume with no loss of generality that k_0 is transcendental and the other k_i's are all rational numbers, and therefore, $C_{rr} \neq 0$.) Thus, (2.2) can be written as a polynomial in $\{\sum_{i=1}^{n} a_i^j : j = 1,..., r-1\} + C_{rr} \cdot \sum_{i=1}^{n} a_i^r$. It follows after equating coefficients of x^{nt+r}, $1 \leqslant r \leqslant n$, that

$$\sum_{i=1}^{n} a_i^r = \sum_{i=1}^{n} b_i^r, \quad r = 1, 2,..., n.$$

Using Lemma 2.2, it follows that $\prod_{i=1}^{n}(x-a_i)=\prod_{i=1}^{n}(x-b_i)$. This means that $\{a_1,..., a_n\}$ is a rearrangement of $\{b_1,..., b_n\}$.

B. In (A) above we assumed that $C_{rr} \neq 0$ $(1 \leqslant r \leqslant n)$. Here we make a different assumption. We again assume the identity (2.1). Now we make the following assumption:

> The function $f(x)=F'(x)/F(x)$ can be exanded in an INFINITE power series about 0 so that $f(x)=f(0)+xf'(0)+(x^2/2)f''(0)+\cdots$, $-a<x<a$.

Note that the above assumption holds for Cauchy distributions. We may also remark that when $[F'(x)/F(x)]$ is a polynomial, then $F(x)$ is of the form exp (polynomial) and for such functions, the identification of parameters is not possible based only on the distribution of the maximum random variable. (See e.g., [1, p. 240 (2.18)] for a counterexample.)

Now we take the logarithm of both sides in (2.1) and differentiate with respect to x. Then we have, for $-a<x<a$,

$$\sum_{i=1}^{n} a_i[F'(a_ix)/F(a_ix)]=\sum_{i=1}^{s} b_i[F'(b_ix)/F(b_ix)].$$

Using our above assumption, we can now write

$$\sum_{i=1}^{n}\left(\sum_{m=0}^{\infty} a_i^{m+1}x^m[f^{(m)}(0)/m!]\right)=\sum_{i=1}^{s}\left(\sum_{m=0}^{\infty} b_i^{m+1}x^m[f^{(m)}(0)/m!]\right)$$

which can be rewritten as

$$\sum_{m=0}^{\infty}\left(\sum_{i=1}^{n} a_i^{m+1}\right)x^m[f^{(m)}(0)/m!]=\sum_{m=0}^{\infty}\left(\sum_{i=1}^{s} b_i^{m+1}\right)x^m[f^{(m)}(0)/m!].$$

Since by assumption the power series for $f(x)$ is an infinite power series, we have by equating coefficients from both sides of the above identity,

$$\sum_{i=1}^{n} a_i^m=\sum_{i=1}^{s} b_i^m \tag{2.3}$$

for infinitely many m. Now notice that F as well as F_i's and G_i's are all distribution functions. Consequently, the a_i's as well as the b_i's are nonnegative. It is then elementary to observe that

$$\left(\sum_{i=1}^{n} a_i^{m_k}\right)^{1/m_k} \to \max\{a_i: 1 \leqslant i \leqslant n\}$$

as the sequence $m_k \to \infty$. Therefore, it follows from (2.3) that

$$\max\{a_i: 1 \leqslant i \leqslant n\} = \max\{b_i: 1 \leqslant i \leqslant s\}.$$

Cancelling these maximal terms from both sides of (2.3) and repeating the same procedure, we obtain after a finite number of steps $n = s$ and $\{a_1, \ldots, a_n\}$, a rearrangement of $\{b_1, \ldots, b_s\}$.

3

A typical non-singular (i.e., with a non-singular covariance matrix) bivariate normal distribution with zero means can be written in the form

$$F(x, y) = (ab\sqrt{1-r^2})/2\pi \cdot \int_{-\infty}^{x} \int_{-\infty}^{y} \exp\{-\tfrac{1}{2}(a^2u^2 - 2abruv + b^2v^2)\}\, du\, dv.$$

Here $s\sqrt{1-r^2} = 1/a$, $t\sqrt{1-r^2} = 1/b$, where r is the correlation, s^2 is the x variance and t^2 is the y variance.

We will now give a complete solution of the Anderson–Ghurye problem (namely, the problem of this paper) for general bivariate normal distributions with zero means. It will also be clear from our proof that the solution is also valid for such distributions with means not necessarily zero.

THEOREM 3.1. *Suppose that $F_1, F_2, \ldots, F_n$ are non-singular bivariate normal distributions with zero means and that at least one of them has a nonzero correlation. Also suppose that*

$$\prod_{i=1}^{n} F_i^{c_i} = 1 \tag{3.1}$$

where each c_i is 1 or -1. Then (3.1) can always be reduced to an equality where the number of factors on the left side of (3.1) is $n - 2m, m \geqslant 1$; moreover, for the factors that simplify out, $\sum c_i = 0$, the summation being over these factors, and the parameters are the same.

Before we prove the theorem, let us make some necessary observations. Taking "log" of both the sides in (3.1) and differentiating partially, we have

$$\sum c[F_x/F] = \sum c[F_y/F] = 0.$$

Differentiating again partially with respect to y, we have

$$\sum_{r \neq 0} c[\{F_{xy}/F\} - \{F_x F_y\}/F^2] = 0 \tag{3.2}$$

since $(\partial/\partial y)\,[F_x/F]=0$ whenever $r=0$. A typical bivariate normal cdf F has the following six properties:

(i) $$F_{xy}=(ab\sqrt{(1-r^2)}/2\pi)\cdot e^{-(1/2)(a^2x^2-2rabxy+b^2y^2)}. \tag{3.3}$$

(ii) $$F_x=(a\sqrt{(1-r^2)}/\sqrt{2\pi})\cdot e^{-(1/2)a^2(1-r^2)x^2}\cdot N(by-arx), \tag{3.4}$$

$$F_y=\frac{b\sqrt{(1-r^2)}}{\sqrt{2\pi}}\cdot e^{-(1/2)b^2(1-r^2)y^2}\cdot N(ax-bry), \tag{3.5}$$

where $N(x)$ is the standard normal cdf.

(iii) Write $T(x)\equiv 1-N(x)=N(-x)$. Then,

$$T(x)\sim\frac{1}{\sqrt{2\pi}x}\cdot e^{-(1/2)x^2}\qquad(\text{as } x\to\infty). \tag{3.6}$$

(iv) Two terms of the form

$$\exp\{-\tfrac{1}{2}(A^2x^2+Cxy+B^2y^2)\} \tag{3.7}$$

are either identical or one dominates the other. (If $f_1/f_2\to 0$ as $x\to\infty$, $y\to\infty$, and $x>>y$, then we say that f_2 dominates f_1.)

(v) If $r>0$, then F_{xy} dominates F_xF_y. (3.8)

Proof of (v). From (3.4), we have

$$F_x=\frac{a\sqrt{1-r^2}}{\sqrt{2\pi}}\cdot e^{-(1/2)a^2(1-r^2)x^2}[1-N(arx-by)]$$

$$\sim\frac{a\sqrt{1-r^2}}{2\pi}\,e^{-(1/2)a^2(1-r^2)x^2}\cdot\frac{e^{-(1/2)(arx-by)^2}}{arx-by}\qquad(\text{as } x\to\infty,\ y\to\infty,\ x>>y)$$

and

$$F_y\sim[b\sqrt{1-r^2}/\sqrt{2\pi}]\,e^{-(1/2)b^2(1-r^2)y^2}$$

(as $x\to\infty$, $y\to\infty$ and $x>>y$ since then $N(ax-bry)\to 1$). It is now clear that F_{xy} dominates F_xF_y. ▮

(vi) If $r<0$, then F_xF_y dominates F_{xy}. (3.9)

Proof of (vi). Notice that for $r<0$, we have using (3.4) and (3.5):

$$F_xF_y\sim\frac{ab(1-r^2)}{2\pi}\cdot e^{-(1/2)[a^2(1-r^2)x^2+b^2(1-r^2)y^2]}. \tag{3.10}$$

The assertion is now clear since

$$a^2(1-r^2)x^2+b^2(1-r^2)y^2<a^2x^2+b^2y^2\leqslant a^2x^2-2abrxy+b^2y^2. \quad \blacksquare$$

We are now ready to prove the theorem.

Proof of Theorem 3.1. Let $x\to\infty$, $y\to\infty$, and $x>>y$. Consider the term that dominates all terms in the left-hand side of (3.2) which are different from it in absolute value. Note that there is always such a term by observations (i), (iv), (v), and (vi). Now we consider the two cases.

Case I. $r>0$ for the dominating term. In this case, we divide both sides of (3.2) by the F_{xy} of the dominating term and obtain in the limit $\sum c=0$, where the summation is over the dominating terms. Since the c's are $+1$ or -1, there are an even number of terms which have identical expressions for F_{xy} and this means that for these terms, a, b, and r are the same. In other words, Eq. (3.1) can be reduced as claimed.

Case II (the non-trivial case). $r<0$ for the dominating term. In this case, we divide both sides of (3.2) by the F_xF_y (see (3.10)) of the dominating term and obtain $\sum c=0$, where the summation is over the dominating terms, and for these terms, the quantities $a^2(1-r^2)$ $(=1/s^2)$ and $b^2(1-r^2)$ $(=1/t^2)$ are the same. Thus, the parameter r is still to be taken care of, and the difficulty of the problem is right here. This difficulty can be overcome by the following lemma.

LEMMA. *For $r<0$, we have as $x\to\infty$, $y\to\infty$, and $x>>y$,*

$$\frac{F_xF_y}{F^2}=\frac{ab(1-r^2)}{2\pi}\cdot\frac{e^{-(1/2)[a^2(1-r^2)x^2+b^2(1-r^2)y^2]}}{[1-T(a\sqrt{1-r^2}x)-T(b\sqrt{1-r^2}y)]^2}+o(F_{xy}).$$

Once we have proven this lemma, then looking at the dominating terms (and recalling that $a\sqrt{1-r^2}$ and $b\sqrt{1-r^2}$ are the same for these terms, and also, $\sum c=0$ for these terms), it is clear that over the dominating terms, the summation

$$\sum c\cdot\left(\frac{F_{xy}}{F}-\frac{F_xF_y}{F^2}\right)$$

is reduced to the summation

$$\sum c\cdot\left(\frac{F_{xy}}{F}-o(F_{xy})\right).$$

Thus, in Eq. (3.2), replacing the group of dominating terms by $\sum c\cdot(F_{xy}/F-o(F_{xy}))$, the summation here being over this group of dominating terms, we see that there are now more terms than before in this equation where F_{xy}, rather than $F_x F_y$, dominates. We repeat this process over and over again till in every term of Eq. (3.2), F_{xy} is the dominating part. Now as in Case 1, it is clear that Eq. (3.2) can be reduced as claimed. Therefore, for the completion of the proof, only the lemma is left to be proven.

Proof of the lemma. From (3.4), we have

$$\begin{aligned}
F_x &= \frac{a\sqrt{1-r^2}}{\sqrt{2\pi}} \cdot e^{-(1/2)a^2(1-r^2)x^2}[1 - N(arx - by)] \\
&\sim \frac{a\sqrt{1-r^2}}{\sqrt{2\pi}} e^{-(1/2)a^2(1-r^2)x^2} - \frac{a\sqrt{1-r^2}}{2\pi} \cdot e^{-(1/2)a^2(1-r^2)x^2} \cdot \frac{e^{-(1/2)(by-arx)^2}}{by-arx} \\
&= \frac{a\sqrt{1-r^2}}{\sqrt{2\pi}} \cdot e^{-(1/2)a^2(1-r^2)x^2} - \frac{a\sqrt{1-r^2}}{2\pi} \cdot \frac{e^{-(1/2)(a^2x^2 - 2abrxy + b^2y^2)}}{by-arx} \\
&= \frac{a\sqrt{1-r^2}}{\sqrt{2\pi}} \cdot e^{-(1/2)a^2(1-r^2)x^2} + o(F_{xy}).
\end{aligned}$$

Similarly, we have

$$F_y \sim \frac{b\sqrt{1-r^2}}{\sqrt{2\pi}} \cdot e^{-(1/2)b^2(1-r^2)y^2} + o(F_{xy}).$$

We now estimate F. Observe that

$$\begin{aligned}
F(x, y) &= \frac{ab\sqrt{1-r^2}}{2\pi} \int_{-\infty}^{x} \int_{-\infty}^{y} e^{-1/2(a^2u^2 - 2abruv + b^2v^2)} du\, dv \\
&= \frac{ab\sqrt{1-r^2}}{2\pi} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \int_{y}^{\infty} - \int_{x}^{\infty} \int_{-\infty}^{\infty} + \int_{x}^{\infty} \int_{y}^{\infty} \right].
\end{aligned}$$

The first term is 1. The third term can be written as

$$\begin{aligned}
&\frac{a\sqrt{1-r^2}}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-(1/2)a^2(1-r^2)u^2} du \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(1/2)(bv - aru)^2} b dv \right] \\
&\quad = T(a\sqrt{1-r^2}x).
\end{aligned}$$

Similarly, the second term can be written as $T(b\sqrt{1-r^2}y)$. Now we rewrite the fourth term as

$$\frac{a\sqrt{1-r^2}}{\sqrt{2\pi}}\cdot\int_x^{\infty} e^{-(1/2)a^2(1-r^2)u^2}du\left[\frac{1}{\sqrt{2\pi}}\int_y^{\infty} e^{-(1/2)(bv-aru)^2}bdv\right]$$

$$\leqslant\frac{a\sqrt{1-r^2}}{\sqrt{2\pi}}\cdot\int_x^{\infty} e^{-(1/2)a^2(1-r^2)u^2}T(by-arx)du$$

$$\text{(since for } r<0 \text{ and } u>x,\ T(by-aru)<T(by-arx))$$

$$<\frac{a\sqrt{1-r^2}}{\sqrt{2\pi}}\cdot\frac{e^{-(1/2)(by-arx)^2}}{by-arx}\cdot\int_x^{\infty} e^{-(1/2)a^2(1-r^2)u^2}du$$

$$\sim\frac{e^{-(1/2)(by-arx)^2}}{by-arx}\cdot\frac{e^{-(1/2)a^2(1-r^2)x^2}}{a\sqrt{1-r^2}x}=o(F_{xy}).$$

Thus, it is no loss of generality to write

$$F(x,y)=1-T(a\sqrt{1-r^2}x)-T(b\sqrt{1-r^2}y)+o(F_{xy}).$$

Thus, we can also write

$$\frac{F_xF_y}{F^2}=\frac{(ab(1-r^2)/2\pi)\cdot e^{-(1/2)[a^2(1-r^2)x^2+b^2(1-r^2)y^2]}+o(F_{xy})}{[1-T(a\sqrt{1-r^2}\cdot x)-T(b\sqrt{1-r^2}\cdot y)]^2+o(F_{xy})}.$$

The lemma now follows. ∎

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